Boundary States for a Free Boson Defined on Finite Geometries

Marc-André Lewis*
Centre de recherches mathématiques, Montréal, Canada, and
Laboratoire de Physique Théorique et des Hautes Energies,
Universités Pierre et Marie Curie (Paris VI) et Denis Diderot (Paris VII),
Paris, France

Yvan Saint-Aubin†
Centre de recherches mathématiques and
Dépt. de mathématiques et de statistique,
Université de Montréal, C.P. 6128, succ. centre-ville,
Montréal, Québec, Canada H3C 3J7

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Abstract

Langlands recently constructed a map $\varphi \to |\mathfrak{x}(\varphi)\rangle$ that factorizes the partition function of a free boson on a cylinder with boundary condition given by two arbitrary functions φ_{B_1} and φ_{B_2} in the form $\langle\mathfrak{x}(\varphi_{B1})|q^{L_0+\bar{L}_0}|\mathfrak{x}(\varphi_{B2})\rangle$. We rewrite $|\mathfrak{x}(\varphi)\rangle$ in a compact form, getting rid of technical assumptions necessary in his construction. This simpler form allows us to show explicitly that the map $\varphi \to |\mathfrak{x}(\varphi)\rangle$ commutes with conformal transformations preserving the boundary and the reality condition on the field φ.

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*Email: lewism@cirano.umontreal.ca
†Email: saint@crm.umontreal.ca

1 Introduction

The study of conformal field theories (CFT's) defined on finite geometries was initiated by Cardy in 1986 [3]. Models studied since then include conformally invariant (free or fixed) and integrable (constant magnetic field and Sine-Gordon potential) boundary conditions. Here, we shall consider more general boundary conditions.

We restrict our attention to a simple model of CFT with boundary conditions: the free scalar field defined on a cylinder of finite length and radius. The field on the cylinder will be constrained to take values at the boundary given by two boundary functions φ_{B1} and φ_{B2}, each associated to one extremity of the cylinder. Langlands [2] has calculated the partition function Z for fields satisfying these boundary conditions. We derive here explicit expressions for the boundary states $|\mathfrak{x}(\varphi)\rangle$, *i.e.* we define a map $\mathfrak{x}$ from the space of boundary functions to some suitably chosen Hilbert space $\mathcal{H}$ such that the partition function Z can be written as an inner product in $\mathcal{H}$:

$$Z_{12} = \langle \mathfrak{x}(\varphi_{B1})|\mathcal{O}|\mathfrak{x}(\varphi_{B2})\rangle, \tag{1}$$

where $\mathcal{O}$ is the evolution operator going from one side of the cylinder to the other. Using the isomorphism between the Fock space and its functional representation, we rewrite $|\mathfrak{x}(\varphi)\rangle$ in a simpler, exponential form. We conclude this note by looking at the transformation of boundary states under infinitesimal conformal transformations that preserve the boundary.

2 Notations

The description of free bosons is based on the Heisenberg algebra and its representations. The generators are the creation ($\mathfrak{a}_{-k}$, $k > 0$), annihilation ($\mathfrak{a}_k$, $k > 0$) and central ($\mathfrak{a}_0$) operators, obeying the commutation rule

$$[\mathfrak{a}_n, \mathfrak{a}_m] = n\delta_{n+m,0}. \tag{2}$$

The Fock space $\mathcal{F}_\alpha$ is a highest weight representation. The action of the generators on the highest weight vector $|\alpha\rangle$ is given by:

$$\mathfrak{a}_k|\alpha\rangle = 0, \qquad \forall k > 0, \tag{3}$$

$$\mathfrak{a}_0|\alpha\rangle = \alpha|\alpha\rangle, \tag{4}$$

and physical states are generated by polynomials in the $\mathfrak{a}_{-k}$, $k > 0$. A basis for $\mathcal{F}_\alpha$ is given by the vectors

$$|\alpha; n_1, n_2, \cdots\rangle = \mathfrak{a}_{-1}^{n_1}\mathfrak{a}_{-2}^{n_2}\cdots|\alpha\rangle, \tag{5}$$

where the non-negative integers n_i's are all zero but finitely many. The inner product on $\mathcal{F}_\alpha$ is defined by

$$\begin{aligned}\langle\alpha'; n'_1, n'_2, \cdots|\alpha; n_1, n_2, \cdots\rangle &= \langle\alpha'|\cdots\mathfrak{a}_3^{n'_3}\mathfrak{a}_2^{n'_2}\mathfrak{a}_1^{n'_1}\mathfrak{a}_{-1}^{n_1}\mathfrak{a}_{-2}^{n_2}\mathfrak{a}_{-3}^{n_3}\cdots|\alpha\rangle \\ &= \delta_{\alpha,\alpha'}\left(\prod_{k=1}^{\infty} k^{n_k}\, n_k!\,\delta_{n_k,n'_k}\right).\end{aligned} \tag{6}$$

The elements (5) can thus be easily normalized to form an orthonormal basis.

The Hilbert space of the free boson is the direct sum of tensor products of the form $\mathcal{F}_\alpha \otimes \mathcal{F}_{\bar{\alpha}}$, $\mathcal{F}_\alpha$ and $\mathcal{F}_{\bar{\alpha}}$ characterizing modes in the holomorphic and anti-holomorphic sectors, respectively. States in these tensor products are generated by the action of polynomials in the $\mathfrak{a}_{-k}$ and $\bar{\mathfrak{a}}_{-k}$ on the highest weight vector $|\alpha;\bar{\alpha}\rangle = |\alpha\rangle \otimes |\bar{\alpha}\rangle$. The generators $\mathfrak{a}_k$ are understood to act as $\mathfrak{a}_k \otimes 1$ and the $\bar{\mathfrak{a}}_k$ as $1 \otimes \mathfrak{a}_k$.

Fock spaces are given the structure of a Virasoro module by defining the conformal generators

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} :\mathfrak{a}_{n-m}\mathfrak{a}_m: \qquad n \neq 0 \\ L_0 &= \sum_{n>0}\mathfrak{a}_{-n}\mathfrak{a}_n + \frac{1}{2}\mathfrak{a}_0^2. \end{aligned} \tag{7}$$

The expression for L_0 implies that $\mathcal{F}_\alpha$ is a highest weight module with highest weight $\alpha^2/2$. As will be described in the next section, the boson field is to be compactified on a circle of radius R. The pairs $(\alpha, \bar{\alpha})$ are then restricted to take the values

$$\alpha = \alpha_{u,v} = \left(\frac{u}{2R} + vR\right) \qquad \rightarrow \qquad h_{u,v} = \frac{1}{2}\left(\frac{u}{2R} + vR\right)^2 \tag{8}$$

$$\bar{\alpha} = \bar{\alpha}_{u,v} = \alpha_{u,-v} \qquad \rightarrow \qquad \bar{h}_{u,v} = \frac{1}{2}\left(\frac{u}{2R} - vR\right)^2, \tag{9}$$

with u and v integers and where $h_{u,v}$ and $\bar{h}_{u,v}$ are the values of $L_0 = L_0 \otimes 1$ and $\bar{L}_0 = 1 \otimes L_0$ acting on $\mathcal{F}_{\alpha_{u,v}} \otimes \mathcal{F}_{\bar{\alpha}_{u,v}}$. We will denote $\mathcal{F}_{\alpha_{u,v}}$ by $\mathcal{F}_{(u,v)}$ and $\mathcal{F}_{\bar{\alpha}_{u,v}}$ by $\bar{\mathcal{F}}_{(u,v)}$.

In his calculation, Langlands chose the Virasoro algebra Vir as the fundamental structure. He was able to construct explicitly the map $\mathfrak{x}$ for irreducible Verma modules over Vir. However Verma modules over Vir are reducible for some rational highest weights. (Rational compactification radii R do lead to such highest weights.) It was his suggestion that we look for an alternative definition that would encompass reducible cases. Using the Heisenberg algebra as the basic structure avoids this difficulty and leads as well to an elegant form for $\mathfrak{x}$.

3 Explicit calculation of the partition function

We identify the cylinder with the quotient of the infinite strip $0 < \mathrm{Re}\, w < \ln q$, $0 < q < 1$, by the translations $w \to w + 2\pi i k$, $k \in \mathbb{Z}$. It can be mapped on the annulus $\mathcal{A}$ of center 0, outer radius 1 and inner radius q by the conformal map $z = e^w$. The angle θ of the annular geometry parametrizes both extremities of the cylinder.

The partition function is defined as

$$\int \mathcal{D}\varphi \, e^{-\int_{\mathcal{A}} \mathcal{L}(\varphi) d^2 z} \tag{10}$$

where $\int_{\mathcal{A}}$ denotes the integration over the annulus, and the Lagrangian density is given by

$$\mathcal{L}(\varphi) = \partial_z \varphi \partial_{\bar{z}} \varphi. \tag{11}$$

The usual mode expansion of $\varphi(z, \bar{z})$ is

$$\varphi(z, \bar{z}) = \varphi_0 + a \ln z + b \ln \bar{z} + \sum_{n \neq 0} \left(\varphi_n z^n + \bar{\varphi}_n \bar{z}^n\right).$$

The restriction φ_{B_1} of this field to the inner circle where $z = qe^{i\theta}$ and $\bar{z} = qe^{-i\theta}$ is of the form:

$$\varphi_{B1}(\theta) = \varphi_0 + (a+b)\ln q + i\theta(a-b) + \sum_{k\neq 0} b_k e^{ik\theta}, \qquad b_{-k} = \bar{b}_k$$

and the restriction φ_{B_2} to the outer circle ($z = e^{i\theta}, \bar{z} = e^{-i\theta}$):

$$\varphi_{B2}(\theta) = \varphi_0 + i\theta(a-b) + \sum_{k\neq 0} a_k e^{ik\theta}, \qquad a_{-k} = \bar{a}_k.$$

(The relationship between a_k, b_k and φ_n will be given below.) Since it is the field $e^{i\varphi/R}$ that really matters, φ need not be periodic but should only satisfy the milder requirement $\varphi(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \varphi(z,\bar{z}) + 2\pi v R$, $v \in \mathbb{Z}$. This statement is equivalent to the compactification of the field φ on a circle of radius R and implies that

$$a - b = -ivR, \qquad v \in \mathbb{Z}.$$

The Lagrangian density does not depend on φ_0 and this constant may be set to zero. Therefore only the difference of the constant terms in φ_{B_1} and φ_{B_2} remains. We choose to parametrize this difference by a real number $x \in [0, 2\pi R)$ and an integer $m \in \mathbb{Z}$:

$$-(a+b)\ln q = x + 2\pi m R.$$

The reason for this parametrization is again the compactification of φ: even though the various pairs $(\varphi_{B_1} + 2\pi m R, \varphi_{B_2})$, $m \in \mathbb{Z}$, will give different contributions to the functional integral, they all represent the same restriction of $e^{i\varphi/R}$ at the boundary.

We are interested in computing the partition function $Z(\varphi_{B_1}, \varphi_{B_2}) = Z(x, \{b_k\}, \{a_k\})$ defined as

$$Z(x, \{b_k\}, \{a_k\}) = \int_B \mathcal{D}\varphi\, e^{-\int_{\mathcal{A}} \mathcal{L}(\varphi) d^2z}, \tag{12}$$

where $\int_B$ denotes the integration on the space of functions φ such that the restrictions of $e^{i\varphi/R}$ at the inner and outer boundaries coincide with $e^{i\varphi_{B_1}/R}$ and $e^{i\varphi_{B_2}/R}$. (The dependence on the compactification radius R is always implicit.) The decomposition of the field in a classical part verifying the boundary conditions and fluctuations vanishing at the extremities leads to

$$Z(x,\{b_k\},\{a_k\}) = \Delta^{-1/2} Z_{\text{class}}(x,\{b_k\},\{a_k\}). \tag{13}$$

The factor Δ is the ζ-regularization of the determinant for the annulus and is known to be (see for example [2, 4]):

$$\Delta^{-1/2} = (-i\tau)^{-1/2}\eta^{-1}(\tau), \qquad \text{with } q = e^{i\pi\tau}, \tag{14}$$

where $\eta(\tau) = e^{i\pi\tau/12}\prod_{m=1}^{\infty}(1-e^{2im\pi\tau})$ is the Dedekind η function. The factor Z_{class} is the integration (sum) over all classical solutions compatible with the boundary conditions in the above sense. To obtain Z_{class} we solve the classical equations ($\partial_z\partial_{\bar z}\varphi = 0$) with boundary conditions given by $(\varphi_{B_1},\varphi_{B_2})$. The condition at the outer circle ($z = e^{i\theta}, \bar z = e^{-i\theta}$) is $\varphi_n + \bar\varphi_{-n} = a_n$ and that at the inner one ($z = qe^{i\theta}, \bar z = qe^{-i\theta}$) is $q^n\varphi_n + q^{-n}\bar\varphi_{-n} = b_n$. The solution can be written as the sum

$$\varphi = a\ln z + b\ln\bar z + \tilde\varphi_1 + \tilde\varphi_2 \tag{15}$$

where the two function $\tilde\varphi_1$ and $\tilde\varphi_2$ are harmonic inside the annulus and take respectively the values φ_{B1} and 0 on the inner boundary and the values 0 and φ_{B2} on the outer one. These functions are

$$\tilde\varphi_1(z,\bar z) = \sum_{k\neq 0}\frac{b_k}{q^k - \frac{1}{q^k}}(z^k - \bar z^{-k}),$$
$$\tilde\varphi_2(z,\bar z) = \sum_{k\neq 0}\frac{a_k}{\frac{1}{q^k} - q^k}\left(\left(\frac{z}{q}\right)^k - \left(\frac{\bar z}{q}\right)^{-k}\right).$$

Hence the classical solution φ is completely determined by the data $(x,\{b_k\},\{a_k\})$ up to the two integers $m, v \in \mathbb{Z}$ that determine a and b. The factor Z_{class} is consequently the sum:

$$\sum_{m,v\in\mathbb{Z}} e^{\mathcal{L}(\varphi_{(m,v)})}$$

where $\varphi_{(m,v)}$ is the solution (15) with $-(a+b)\ln q = x + 2\pi mR$ and $a - b = -ivR$.

Using this expression and the Poisson summation formula on the index m, Langlands [2] computed the desired partition function as the product

$$Z(x,\{b_k\},\{a_k\}) = \Delta^{-1/2} Z_1(x) Z_2(\{b_k\},\{a_k\}) \tag{16}$$

where

$$Z_1(x) = \sum_{u,v\in\mathbb{Z}} e^{iux/R} q^{\frac{u^2}{4R^2}+v^2R^2} = \sum_{u,v\in\mathbb{Z}} e^{ix(\alpha_{u,v}+\bar{\alpha}_{u,v})}\, q^{h_{u,v}+\bar{h}_{u,v}} \tag{17}$$

and

$$Z_2(\{b_k\},\{a_k\}) = \prod_{k=1}^{\infty} \exp\left(-2k\left(\frac{1+q^{2k}}{1-q^{2k}}(a_k a_{-k}+b_k b_{-k}) - \frac{2q^k}{1-q^{2k}}(a_k b_{-k}+b_k a_{-k})\right)\right). \tag{18}$$

4 Explicit form of the boundary states

In this section we rewrite (16) as a sum over $u, v \in \mathbb{Z}$ of terms of the form

$$Z^{(u,v)}(x,\{b_k\},\{a_k\}) = \langle \mathfrak{x}^{(u,v)}(\varphi_{B1})| q^{L_0+\bar{L}_0} |\mathfrak{x}^{(u,v)}(\varphi_{B2})\rangle, \tag{19}$$

with $|\mathfrak{x}^{(u,v)}(\varphi)\rangle \in \mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)}$ and where

$$Z^{(u,v)}(x,\{b_k\},\{a_k\}) = \Delta^{-\frac{1}{2}} e^{iux/R} q^{\frac{u^2}{4R^2}+v^2R^2} Z_2(\{b_k\},\{a_k\}).$$

The goal is therefore to find a map $\mathfrak{x}^{(u,v)}$ such that (19) holds. To do so we will first set $Z_2(\{b_k\},\{a_k\})$ in the form

$$Z_2(\{b_k\},\{a_k\}) = \prod_{k=1}^{\infty} \sum_{m,n} B^k_{m,n} q^{k(m+n)} A^k_{m,n} \tag{20}$$

where $A^k_{m,n} = A^k_{m,n}(a_k, a_{-k})$ and $B^k_{m,n} = B^k_{m,n}(b_k, b_{-k})$ are functions of only two variables. With $Q = q^k$, $a_\pm = 2i\sqrt{k} a_{\pm k}$ and $b_\pm = -2i\sqrt{k} b_{\pm k}$ for $k \geq 1$, the terms $\Delta^{-1/2} Z_2$ of (16) are a product over k of

$$e^{a_+a_-/2} e^{b_+b_-/2} \frac{e^{(a_+a_-Q^2+a_+b_-Q+b_+a_-Q+b_+b_-Q^2)/(1-Q^2)}}{1-Q^2}, \tag{21}$$

up to a constant depending only on τ. The factors in front are clearly factorizable and can be absorbed in the definition of $A^k_{m,n}$ and $B^k_{n,m}$. The remaining mixed term can be developped as a power series in Q:

$$\sum_{i,j,k,l=0}^{\infty} \frac{(a_+a_- + b_+b_-)^i}{i!} \frac{(a_+b_-)^j(b_+a_-)^k}{j!k!} \frac{(i+j+k+l)!}{(i+j+k)!l!} Q^{2i+j+k+2l}. \tag{22}$$

To achieve the form (20), the coefficient of the term $q^{k(m+n)}$ (i.e. of $Q^{2i+j+k+2l}$ with $2i + j + k + 2l = m + n$) in the above expression has to be the product of two functions, one of (a_k, a_{-k}), the other of (b_k, b_{-k}). We concentrate on the terms with $j \geq k$ and denote by $S_{m,n}$ the factor of $(a_+b_-)^{m-n}Q^{m+n}$ with $j - k = m - n$. The terms with $j < k$ are treated similarly. With the use of $x = a_+a_-$ and $y = b_+b_-$, $S_{m,n}$ can be written as

$$S_{m,n}(x,y) = \sum_{i+k+l=n} \frac{(x+y)^i(xy)^k}{i!(m-n+k)!k!} \frac{(m+k)!}{(m-n+i+2k)!l!}.$$

This function is clearly symmetric in x and y. Define

$$R_{m,n}(x) \equiv S_{m,n}(x,0)$$

and

$$T_{m,n} \equiv R_{m,n}(0) = S_{m,n}(0,0).$$

Casting (22) in the form (20) will only be possible if

$$S_{m,n}(x,y)T_{m,n} = R_{m,n}(x)R_{m,n}(y). \tag{23}$$

That this condition is verified is highly non-trivial. It was found in [2] that it is, although in disguised form, the Saalschütz identity [9]. We thus have the factorization if we define

$$A^k_{m,n}(a_k, a_{-k}) = \frac{R_{m,n}(x)}{\sqrt{T_{m,n}}} a_+^{m-n} e^{a_+a_-/2}, \qquad \text{if } m \geq n \tag{24}$$

where

$$R_{m,n} = \sum_{i=0}^{n} \frac{m!x^i}{i!(m-n)!(m-n+i)!(n-i)!}, \qquad m \geq n,$$

and

$$T_{m,n} = \frac{m!}{n!((m-n)!)^2}, \qquad m \geq n.$$

This expression for $R_{m,n}$ shows that it is related to the n-th Laguerre polynomial of the $(m-n)$-th kind by

$$R_{m,n}(x) = \frac{L_n^{(m-n)}(-x)}{(m-n)!}, \qquad m \geq n. \tag{25}$$

A similar calculation leads to

$$R_{m,n}(x) = \frac{L_m^{(n-m)}(-x)}{(n-m)!}, \qquad n \geq m. \tag{26}$$

Going back to the initial notation, we finally get the desired form with:

$$A^k_{m,n}(a_k, a_{-k}) = \begin{cases} (2i\sqrt{k}a_k)^{m-n}\sqrt{\frac{n!}{m!}}e^{-2k|a_k|^2}L_n^{(m-n)}(4k|a_k|^2) & m \geq n \\ (-2i\sqrt{k}a_{-k})^{n-m}\sqrt{\frac{m!}{n!}}e^{-2k|a_k|^2}L_m^{(n-m)}(4k|a_k|^2) & n \geq m, \end{cases} \tag{27}$$

and $B^k_{m,n}(b_k, b_{-k}) = \overline{A^k_{m,n}(b_{-k}, b_k)}$. (An orientation on the boundary must be chosen to define the map $\mathfrak{r}$. For example moving in the positive direction of the parameter θ should put the cylinder at one's left. This explains the interchange $b_k \leftrightarrow b_{-k}$ in the functions B.)

It is now straightforward to define the map from the boundary conditions to the Hilbert space. We have just shown that the contribution of the (u, v) sector to the partition function can be written as

$$Z^{(u,v)}(x, \{b_k\}, \{a_k\}) = q^{h_{u,v}+\bar{h}_{u,v}} e^{ix(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_{k\in\mathbb{N}} \sum_{m,n=0}^{\infty} B^k_{m,n} q^{k(m+n)} A^k_{m,n}.$$

(Note that both sectors (u, v) and $(u, -v)$ contribute the same quantity to Z. There seems therefore to be a freedom to attach (u, v) to either $\mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)}$ or $\mathcal{F}_{(u,-v)} \otimes \bar{\mathcal{F}}_{(u,-v)}$. This choice is resolved in the next section.) Using the fact that

$$\langle u, v| \frac{(\mathfrak{a}^m_k \otimes \mathfrak{a}^n_k)\,(\mathfrak{a}^{m'}_{-k'} \otimes \mathfrak{a}^{n'}_{-k'})}{\sqrt{k^{m+n}k'^{m'+n'}m!n!m'!n'!}} |u, v\rangle = \delta_{k,k'}\delta_{m,m'}\delta_{n,n'},$$

we get

$$Z^{(u,v)}(x,\{b_k\},\{a_k\}) = \tag{28}$$

$$= q^{h_{u,v}+\bar{h}_{u,v}} e^{ix(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_{k,k'} \sum_{m,m',n,n'} B^k_{m,n} q^{k(m+n)} A^{k'}_{m',n'} \delta_{k,k'} \delta_{m,m'} \delta_{n,n'} \tag{29}$$

$$= \langle \mathfrak{x}^{(u,v)}(x_1,\{b_k\}) | q^{L_0 \oplus \bar{L}_0} | \mathfrak{x}^{(u,v)}(x_2,\{a_k\}) \rangle \tag{30}$$

where

$$|\mathfrak{x}^{(u,v)}(x_2,\{a_k\})\rangle = e^{ix_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_{k=1}^{\infty} \sum_{m,n=0}^{\infty} A^k_{m,n}(a_k, a_{-k}) \frac{\mathfrak{a}^m_{-k} \otimes \mathfrak{a}^n_{-k}}{\sqrt{k^{m+n} m! n!}} |u,v\rangle. \tag{31}$$

We have reintroduced, somewhat arbitrarily, the constant term x_2 in φ_{B2}. Again, only the difference $x = x_2 - x_1$ between the constant term x_2 in φ_{B2} and x_1 in φ_{B1} has a physical meaning. We now have an explicit form for the map $\mathfrak{x}$.

The vector $|\mathfrak{x}^{(u,v)}(x_2,\{a_k\})\rangle$ can be cast into a simpler form. With the help of the following recursion identities

$$(n+1) L^{(m-(n+1))}_{n+1}(x) - [x\partial_x - x + (m-n)] L^{(m-n)}_n(x) = 0, \qquad m-1 \geq n \geq 0$$

and

$$L^{((m+1)-n)}_n(x) + [\partial_x - 1] L^{(m-n)}_n(x) = 0, \qquad m \geq n \geq 0,$$

we can prove by induction on both indices that

$$A^k_{m,n} = \frac{(\partial_- + \frac{1}{2}a_+)^m (\partial_+ + \frac{1}{2}a_-)^n}{\sqrt{m!n!}} e^{a_+ a_-/2}, \tag{32}$$

where, we recall, $a_\pm = 2i\sqrt{k} a_k$ and $\partial_\pm = \frac{\partial}{\partial a_\pm}$. Defining $\alpha_k = \frac{i}{2}(-\partial_{-k} + 2k a_k)$, $\bar{\alpha}_k = \frac{i}{2}(-\partial_k + 2k a_{-k})$ and $\Omega_k = A^k_{0,0} = e^{a_+ a_-/2} = e^{-2k|a_k|^2}$, we get

$$A^k_{m,n} = \frac{\alpha^m_k \bar{\alpha}^n_k}{\sqrt{k^{m+n} m! n!}} \Omega_k. \tag{33}$$

The correspondence $\mathfrak{a}_{-k} \leftrightarrow i\alpha_k$ and $\bar{\mathfrak{a}}_{-k} \leftrightarrow i\bar{\alpha}_k$ induces an isomorphism with a subalgebra of the Heisenberg algebra since the α_k's and $\bar{\alpha}_k$'s satisfy the commutation rules:

$$[\alpha_n, \alpha_m] = -n\delta_{n+m,0} \qquad [\bar{\alpha}_n, \bar{\alpha}_m] = -n\delta_{n+m,0}$$

$$[\alpha_n, \bar{\alpha}_m] = 0.$$

If $|\alpha_{uv}\rangle \otimes |\bar{\alpha}_{u,v}\rangle$ is identified with $\Omega = \prod_k \Omega_k$ and α_0 (resp. $\bar{\alpha}_0$) is defined as acting by multiplication by $\alpha_{u,v}$ (resp. $\bar{\alpha}_{u,v}$), this correspondence can then be extended to an isomorphism of Heisenberg modules. Since the $\mathfrak{a}_{-k}$'s and α_k's, $k > 0$, all commute with one another, we are able to write down an exponential form for the boundary state:

$$\begin{aligned}
|\mathfrak{x}^{(u,v)}(x_2, \{a_k\})\rangle &= e^{ix_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_k \left\{ \sum_{m,n} \frac{(\alpha_k \mathfrak{a}_{-k})^m (\bar{\alpha}_k \bar{\mathfrak{a}}_{-k})^n}{m!n!k^m k^n} \Omega_k \right\} |u, v\rangle \\
&= e^{ix_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_k \left\{ e^{\alpha_k \mathfrak{a}_{-k}/k} e^{\bar{\alpha}_k \bar{\mathfrak{a}}_{-k}/k} \right\} \Omega |u, v\rangle \\
&= e^{ix_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_k \left\{ e^{(\alpha_k \mathfrak{a}_{-k} + \bar{\alpha}_k \bar{\mathfrak{a}}_{-k})/k} \right\} \Omega |u, v\rangle \\
&= e^{ix_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} e^{\sum_{k\in\mathbb{N}} (\alpha_k \mathfrak{a}_{-k} + \bar{\alpha}_k \bar{\mathfrak{a}}_{-k})/k} \Omega |u, v\rangle. \qquad (34)
\end{aligned}$$

Up to the factor $(-i\tau)^{-1/2} e^{-i\pi\tau/12}$ the partition function takes the following form:

$$Z(x, \{b_k\}, \{a_k\}) = \langle \mathfrak{x}(x_1, \{b_k\}) | q^{L_0 + \bar{L}_0} | \mathfrak{x}(x_2, \{a_k\}) \rangle, \qquad (35)$$

in which we have defined

$$|\mathfrak{x}(x_2, \{a_k\})\rangle = e^{ix_2(\mathfrak{a}_0 + \bar{\mathfrak{a}}_0)} e^{\sum_{k=1}^{\infty} (\alpha_k \mathfrak{a}_{-k} + \bar{\alpha}_k \bar{\mathfrak{a}}_{-k})/k} \Omega |\Lambda\rangle \qquad (36)$$

$$|\Lambda\rangle = \bigoplus_{u,v} |u, v\rangle, \qquad (37)$$

where the operators $\mathfrak{a}_0 = \mathfrak{a}_0 \otimes 1$ and $\bar{\mathfrak{a}}_0 = 1 \otimes \mathfrak{a}_0$ act as the identity times $\alpha_{u,v}$ and $\bar{\alpha}_{u,v}$ on $|u, v\rangle = |\alpha_{u,v}\rangle \otimes |\bar{\alpha}_{u,v}\rangle$. The boundary states $|\mathfrak{x}(x_2, \{a_k\})\rangle$ belong to the direct sum of Fock spaces $\bigoplus_{u,v} \mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)}$ or, more precisely, to the sum $\bigoplus_{u,v} (\mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)})^c$ of some completions that contains formal series like (36). In the next section, it will turn out to be useful to include the x-dependence in Ω, which will then be noted $\Omega_{u,v}$, to highlight its sector:

$$\Omega_{u,v} = e^{ix(\alpha_{u,v}+\bar{\alpha}_{u,v})} \Omega. \qquad (38)$$

5 Conformal Transformations and Boundary States

Having found an explicit and concise form for the boundary states, we can now study their properties under conformal transformations. Let g be an infinitesimal conformal transformation that leaves the boundary unchanged and G the corresponding element in the Virasoro algebra. The purpose of this section is to show that the action of g on the boundary condition φ and that of G on $|\mathfrak{x}(\varphi)\rangle$ commute:

$$|\mathfrak{x}(g\varphi)\rangle = G|\mathfrak{x}(\varphi)\rangle. \tag{39}$$

We first discuss the actions g and G and the correspondence between them.

One can easily convince oneself that the only infinitesimal conformal transformations that preserve the center and radius of a circle in the complex plane are linear combinations of

$$(l_p - \bar{l}_{-p}), \qquad p \in \mathbb{Z}, \tag{40}$$

where the conformal generators l_p and $\bar{l}_p$ are defined as $l_p = -z^{p+1}\partial_z$ and $\bar{l}_p = -\bar{z}^{p+1}\partial_{\bar{z}}$. Note that the subalgebra $\oplus_{p\in\mathbb{Z}}\mathbb{C}(L_p - \bar{L}_{-p}) \subset \mathrm{Vir}\otimes\overline{\mathrm{Vir}}$ is centerless and the mapping defined by $(l_p - \bar{l}_{-p}) \to (L_p - \bar{L}_{-p})$ of the boundary preserving conformal transformation into $\mathrm{Vir}\otimes\overline{\mathrm{Vir}}$ is an isomorphism. However the tranformations $(l_p - \bar{l}_{-p}), p \neq 0$, do not preserve the reality condition imposed on the boundary functions. The generators $(l_p + \bar{l}_p)$ and $i(l_p - \bar{l}_p)$ do. Both reality and geometry preserving conditions are therefore satisfied by the infinitesimal transformations

$$g_0^{(1)} = 1 + i\epsilon\left\{l_0 - \bar{l}_0\right\} \tag{41}$$

$$g_p^{(1)} = 1 + i\epsilon\left\{(l_p + l_{-p}) - (\bar{l}_p + \bar{l}_{-p})\right\}, \qquad p > 0 \tag{42}$$

and

$$g_p^{(2)} = 1 + \epsilon\left\{(l_p - l_{-p}) + (\bar{l}_p - \bar{l}_{-p})\right\}, \qquad p > 0. \tag{43}$$

We shall show that (39) holds if the $g_p^{(i)}$'s are defined as above and the corresponding $G_p^{(i)}$'s are taken to be

$$G_0^{(1)} = 1 + i\epsilon\{L_0 - \bar{L}_0\}, \tag{44}$$

$$G_p^{(1)} = 1 + i\epsilon\left\{(L_p + L_{-p}) - (\bar{L}_p + \bar{L}_{-p})\right\} \tag{45}$$

and

$$G_p^{(2)} = 1 + \epsilon\left\{(L_p - L_{-p}) + (\bar{L}_p - \bar{L}_{-p})\right\}. \tag{46}$$

Since, for $p \neq 0$, we have

$$[g_p^{(1)} - 1, g_0^{(1)} - 1] = -\epsilon p(g_p^{(2)} - 1),$$

the property for the second family of transformations follows directly if it is proven to be true for the first one. The action of G in the rhs of (39) is simply left-multiplication. On the lhs, the action is defined as usual by $(g\varphi)(z,\bar{z}) = \varphi \circ g^{-1}(z,\bar{z})$. We first study the case $p > 0$. For $p = 0$, the particularity of the Sugawara construction will modify the analysis. We will end this section by examining this case.

Let us first compute $|\mathfrak{x}(g_p\varphi)\rangle$ with $g_p = g_p^{(1)}$. Note that, due to the use of the Poisson summation formula to obtain (17), the constant m in $-(a+b)\ln q = x + 2\pi mR$ is not anymore well-defined in the sector (u,v). However the difference $(a-b)$ still is. Only $a-b$ will appear in the variation $g_p\varphi$. As observed in the previous paragraph, the two contributions $Z^{(u,v)}$ and $Z^{(u,-v)}$ are equal. It turns out that equation (39) holds when the functions φ with a given v are mapped into the sectors $\mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)}$, $u \in \mathbb{Z}$. (For the other choice $\mathcal{F}_{(u,-v)} \otimes \bar{\mathcal{F}}_{(u,-v)}$, the actions g and G fail to commute.) The function on the boundary must have the form

$$\varphi(\theta) = x + vR\theta + \sum_{k>0}(a_k e^{ik\theta} + a_{-k}e^{-ik\theta})$$

or, equivalently

$$\varphi(z,\bar{z}) = x + (a\ln z + b\ln\bar{z}) + \sum_{k>0}(a_k z^k + \bar{a}_k\bar{z}^k)$$

with $z = e^{i\theta}$ and $\bar{z} = e^{-i\theta}$ and the reality condition $a_{-k} = \bar{a}_k$. A direct calculation gives

$$g_p\varphi = \tilde{x} + vR\theta + \sum_{k>0} c_k e^{ik\theta} + \bar{c}_k e^{-ik\theta}$$

where

$$c_k = a_k + i\epsilon\left((k+p)a_{k+p} + (k-p)a_{k-p}\right) + \epsilon vR\delta_{k,p} \tag{47}$$

$$\bar{c}_k = \bar{a}_k - i\epsilon\left((k+p)\bar{a}_{k+p} + (k-p)\bar{a}_{k-p}\right) + \epsilon vR\delta_{k,p} \tag{48}$$

$$\tilde{x} = x + i\epsilon p(a_p - \bar{a}_p). \tag{49}$$

One can see that the reality condition imposed on φ is indeed preserved. Since

$$|\mathfrak{x}^{(u,v)}(\tilde{x}, \{c_k\})\rangle = \left(e^{\sum_{k>0}(\alpha_k \mathfrak{a}_{-k} + \bar{\alpha}_k \bar{\mathfrak{a}}_{-k})/k}\Omega_{u,v}\right)\Big|_{g_p\varphi}|u,v\rangle \tag{50}$$

$$= e^{\sum_{k>0}(\tilde{\alpha}_k \mathfrak{a}_{-k} + \tilde{\bar{\alpha}}_k \bar{\mathfrak{a}}_{-k})/k}\tilde{\Omega}_{u,v}|u,v\rangle \tag{51}$$

where we have defined

$$\tilde{\alpha}_k = \frac{i}{2}(-\frac{\partial}{\partial \bar{c}_k} + 2kc_k), \tag{52}$$

$$\tilde{\bar{\alpha}}_k = \frac{i}{2}(-\frac{\partial}{\partial c_k} + 2k\bar{c}_k), \tag{53}$$

$$\tilde{\Omega}_{u,v} = e^{i\tilde{x}(\alpha_{u,v} + \bar{\alpha}_{u,v})}e^{-2\sum_{k>0} kc_k\bar{c}_k}, \tag{54}$$

a first step is to express $\tilde{\alpha}_k$, $\tilde{\bar{\alpha}}_k$, and $\tilde{\Omega}_{u,v}$ in terms of x and the a_k's. This can be easily achieved. The expressions for c_k and $\bar{c}_k$ given above can be inverted in order to obtain closed form expressions for a_k and $\bar{a}_k$. It is then a simple exercise to show that

$$\tilde{\alpha}_k = \frac{i}{2}(-\frac{\partial}{\partial \bar{c}_k} + 2kc_k) = \begin{cases} \alpha_k + i\epsilon k(\alpha_{k+p} + \alpha_{k-p}), & k \neq p, \\ \alpha_p + i\epsilon p(\alpha_{2p} + \alpha_{uv}), & k = p, \end{cases}$$

$$\tilde{\bar{\alpha}}_k = \frac{i}{2}(-\frac{\partial}{\partial c_k} + 2k\bar{c}_k) = \begin{cases} \bar{\alpha}_k - i\epsilon k(\bar{\alpha}_{k+p} + \bar{\alpha}_{k-p}), & k \neq p, \\ \bar{\alpha}_k - i\epsilon p(\bar{\alpha}_{2p} + \bar{\alpha}_{uv}), & k = p. \end{cases}$$

Using these expressions, the functional $\tilde{\Omega}_{u,v} = \Omega_{u,v}\big|_{g_p\varphi}$ can be expressed in terms of the original variables. A careful treatment of the infinite sums leads to

$$\tilde{\Omega}_{u,v} = \left(1 + i\epsilon\Big(\frac{1}{2}\sum_{0<k<p}(\alpha_{p-k}\alpha_k - \bar{\alpha}_{p-k}\bar{\alpha}_k) + \alpha_p\alpha_{u,v} - \bar{\alpha}_p\bar{\alpha}_{u,v}\Big)\right)\Omega_{u,v}. \tag{55}$$

Finally we can rewrite $|\mathfrak{x}^{(u,v)}(g_p\varphi)\rangle$ to first order in ϵ as

$$
\begin{aligned}
|\mathfrak{x}^{(u,v)}(g_p\varphi)\rangle &= e^{\sum_k(\tilde{\alpha}_k\mathfrak{a}_{-k}+\tilde{\bar{\alpha}}_k\bar{\mathfrak{a}}_{-k})/k}\tilde{\Omega}_{u,v}|u,v\rangle \\
&= e^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k} \\
&\times\left(1+i\epsilon\sum_{k>0}\Big((\alpha_{k+p}+\alpha_{k-p})\mathfrak{a}_{-k}-(\bar{\alpha}_{k+p}+\bar{\alpha}_{k-p})\bar{\mathfrak{a}}_{-k}\Big)\right. &(56)\\
&\left.+i\epsilon\Big(\alpha_p\mathfrak{a}_0-\bar{\alpha}_p\bar{\mathfrak{a}}_0+\frac{1}{2}\sum_{0<k<p}(\alpha_{p-k}\alpha_k-\bar{\alpha}_{p-k}\bar{\alpha}_k)\Big)\right)\Omega_{u,v}|u,v\rangle. &(57)
\end{aligned}
$$

We now turn our attention to the rhs of (39), namely $G_p|\mathfrak{x}(\varphi)\rangle$. It is convenient to introduce the operator $u^k_{m,n}$ defined as

$$u^k_{m,n}=\frac{\mathfrak{a}^m_{-k}\otimes\mathfrak{a}^n_{-k}}{\sqrt{k^{m+n}m!n!}}. \tag{58}$$

It is such that

$$\langle u,v|u^{k\,\dagger}_{m,n}|\mathfrak{x}(\varphi)\rangle=A^k_{m,n}(\varphi)\frac{\Omega_{u,v}}{\Omega_k}, \tag{59}$$

where, we recall, $\Omega_k=e^{-2k|a_k|^2}$. Moreover

$$\left(u^{k'}_{m',n'}|u,v\rangle\right)^\dagger\left(u^k_{m,n}|u,v\rangle\right)=\delta_{k,k'}\delta_{m,m'}\delta_{n,n'} \tag{60}$$

and

$$\prod_{k=1}^{\infty}\sum_{m,n=0}^{\infty}u^k_{m,n}|u,v\rangle\langle u,v|u^{k\,\dagger}_{m,n} \tag{61}$$

acts as the identify on $\mathcal{F}_{(u,v)}\otimes\bar{\mathcal{F}}_{(u,v)}$.

¿From the Sugawara construction

$$L_p=\frac{1}{2}\sum_{k\in\mathbb{Z}}:\mathfrak{a}_{p-k}\mathfrak{a}_k:,\qquad p\neq 0, \tag{62}$$

we see that

$$[L_p,\mathfrak{a}^m_k]=-mk\mathfrak{a}^{m-1}_k\mathfrak{a}_{k+p} \tag{63}$$

with similar equations for the anti-holomorphic sector. Also, since $\langle u,v|\mathfrak{a}_k = 0$ if $k<0$,

$$\langle u,v|L_p \quad = \quad \langle u,v|\Big(\frac{1}{2}\sum_{0<k<p}\mathfrak{a}_k\mathfrak{a}_{p-k} + \mathfrak{a}_p\mathfrak{a}_0\Big). \tag{64}$$

For the same reason, the operator L_{-p} annihilates $\langle u,v|$ when acting from the right. We thus have

$$\begin{aligned}\langle u,v|u^{k\,\dagger}_{m,n}L_p \quad &= \quad \langle u,v|\left([u^{k\,\dagger}_{m,n},L_p] + L_p u^{k\,\dagger}_{m,n}\right)\\ &= \quad \langle u,v|\left(mk\frac{(\mathfrak{a}_k^{m-1}\mathfrak{a}_{k+p})\otimes\mathfrak{a}_k^n}{\sqrt{k^{m+n}m!n!}} + L_p u^{k\,\dagger}_{m,n}\right)\end{aligned} \tag{65}$$

and

$$\begin{aligned}\langle u,v|u^{k\,\dagger}_{m,n}L_{-p} \quad &= \quad \langle u,v|\left([u^{k\,\dagger}_{m,n},L_{-p}]\right)\\ &= \quad \langle u,v|\left(mk\frac{(\mathfrak{a}_k^{m-1}\mathfrak{a}_{k-p})\otimes\mathfrak{a}_k^n}{\sqrt{k^{m+n}m!n!}}\right).\end{aligned} \tag{66}$$

With these two relations and their equivalent in the anti-holomorphic sector, we can compute $\langle u,v|u^{k\,\dagger}_{m,n}G_p$:

$$\begin{aligned}\langle u,v|u^{k\,\dagger}_{m,n}G_p \quad &= \quad \langle u,v|\left(u^{k\,\dagger}_{m,n} + i\epsilon k\frac{\mathfrak{a}_k^{m-1}\otimes\mathfrak{a}_k^{n-1}}{\sqrt{k^{m+n}m!n!}}(m(\mathfrak{a}_{k+p}+\mathfrak{a}_{k-p})\bar{\mathfrak{a}}_k - n\mathfrak{a}_k(\bar{\mathfrak{a}}_{k+p}+\bar{\mathfrak{a}}_{k-p}))\right)\\ &+\langle u,v|\left(\frac{i\epsilon}{2}\sum_{0<l<p}\mathfrak{a}_l\mathfrak{a}_{p-l} - \bar{\mathfrak{a}}_l\bar{\mathfrak{a}}_{p-l} + \mathfrak{a}_p\mathfrak{a}_0 - \bar{\mathfrak{a}}_p\bar{\mathfrak{a}}_0\right)u^{k\,\dagger}_{m,n}.\end{aligned} \tag{67}$$

We thus get

$$\begin{aligned}\langle u,v|u^{k\,\dagger}_{m,n}G_p|\mathfrak{x}(\varphi)\rangle = &\left(1 + i\epsilon\Big(\frac{1}{2}\sum_{0<l<p}\alpha_l\alpha_{p-l} - \bar{\alpha}_l\bar{\alpha}_{p-l} + \alpha_p\alpha_{u,v} - \bar{\alpha}_p\bar{\alpha}_{u,v}\Big)\right)A^k_{m,n}\frac{\Omega_{u,v}}{\Omega_k}\\ &+ i\epsilon k\frac{\alpha_k^{m-1}\bar{\alpha}_k^{n-1}}{\sqrt{k^{m+n}m!n!}}\Big(m(\alpha_{k+p}+\alpha_{k-p})\bar{\alpha}_k - n\alpha_k(\bar{\alpha}_{k+p}+\bar{\alpha}_{k-p})\Big)\Omega_{u,v}.\end{aligned} \tag{68}$$

Once again $\bar{\alpha}_0$ acts on $\Omega_{u,v}$ by multiplication by $\bar{\alpha}_{u,v}$. Using the completeness relation (61), we can reconstruct $G_p|\mathfrak{x}^{(u,v)}(\varphi)\rangle$. First, summing over m and n gives

$$\begin{aligned}
&\sum_{m,n} \frac{\langle u,v|u^{k\,\dagger}_{m,n} G_p|\mathfrak{x}^{(u,v)}(\varphi)\rangle}{\prod_{l\neq k}\Omega_l}\, u^k_{m,n}|u,v\rangle \\
&= \frac{1}{\prod_{l\neq k}\Omega_l}\left(1+i\epsilon\left(\frac{1}{2}\sum_{0<l<p}\alpha_l\alpha_{p-l}-\bar{\alpha}_l\bar{\alpha}_{p-l}+\alpha_p\mathfrak{a}_0-\bar{\alpha}_p\bar{\mathfrak{a}}_0\right)\right)\left(\sum_{m,n}A^k_{m,n}\prod_{l\neq k}\Omega_l\frac{\mathfrak{a}^m_{-k}\bar{\mathfrak{a}}^n_{-k}}{\sqrt{k^{m+n}m!n!}}\right)|u,v\rangle \\
&+\frac{i\epsilon}{\prod_{l\neq k}\Omega_l}\left(k(\alpha_{k+p}+\alpha_{k-p})\sum_{m\geq 1,n\geq 0} m\alpha_k^{m-1}\bar{\alpha}_k^n\frac{\mathfrak{a}^m_{-k}\bar{\mathfrak{a}}^n_{-k}}{k^{m+n}m!n!}\right)\Omega_{u,v}|u,v\rangle \\
&-\frac{i\epsilon}{\prod_{l\neq k}\Omega_l}\left(k(\bar{\alpha}_{k+p}+\bar{\alpha}_{k-p})\sum_{m\geq 0,n\geq 1} n\alpha_k^{m}\bar{\alpha}_k^{n-1}\frac{\mathfrak{a}^m_{-k}\bar{\mathfrak{a}}^n_{-k}}{k^{m+n}m!n!}\right)\Omega_{u,v}|u,v\rangle.
\end{aligned}$$

The last two terms can be rewritten as

$$\begin{aligned}
&\frac{i\epsilon}{\prod_{l\neq k}\Omega_l}\left((\alpha_{k+p}+\alpha_{k-p})\mathfrak{a}_{-k}\sum_{m\geq 0,n\geq 0}\alpha_k^m\bar{\alpha}_k^n\frac{\mathfrak{a}^m_{-k}\bar{\mathfrak{a}}^n_{-k}}{k^{m+n}m!n!}\right)\Omega_{u,v}|u,v\rangle \\
&-\frac{i\epsilon}{\prod_{l\neq k}\Omega_l}\left((\bar{\alpha}_{k+p}+\bar{\alpha}_{k-p})\bar{\mathfrak{a}}_{-k}\sum_{m\geq 0,n\geq 0}\alpha_k^m\bar{\alpha}_k^n\frac{\mathfrak{a}^m_{-k}\bar{\mathfrak{a}}^n_{-k}}{k^{m+n}m!n!}\right)\Omega_{u,v}|u,v\rangle. \qquad (69)
\end{aligned}$$

Putting all this together, we finally have

$$\begin{aligned}
G_p|\mathfrak{x}^{(u,v)}(\varphi)\rangle &= e^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k} \\
&\times\Bigg[1+i\epsilon\sum_{k>0}\Big((\alpha_{k+p}+\alpha_{k-p})\mathfrak{a}_{-k}-(\bar{\alpha}_{k+p}+\bar{\alpha}_{k-p})\bar{\mathfrak{a}}_{-k}\Big) \\
&+i\epsilon\Big(\alpha_p\mathfrak{a}_0-\bar{\alpha}_p\bar{\mathfrak{a}}_0+\frac{1}{2}\sum_{0<k<p}(\alpha_{p-k}\alpha_k-\bar{\alpha}_{p-k}\bar{\alpha}_k)\Big)\Bigg]\,\Omega_{u,v}|u,v\rangle \\
&= |\mathfrak{x}^{(u,v)}(g_p\varphi)\rangle. \qquad (70)
\end{aligned}$$

We have thus established the desired property for $p>0$.

The transformation $g_0^{(1)} = 1+i\epsilon(l_0-\bar{l}_0)$ is nothing but an infinitesimal rotation. The gaussian terms are invariant under these transformations, because Fourier coefficients only pick up a phase. Hence $\Omega_{u,v}|_{g_0^{(1)}\varphi} = (1+i\epsilon vR(\alpha_{u,v}+\bar{\alpha}_{u,v}))\,\Omega_{u,v}|_{\varphi}$. The computation of

$|\mathfrak{x}^{(u,v)}(g_0^{(1)}\varphi)\rangle$ is straightforward and one gets

$$\begin{aligned}
|\mathfrak{x}^{(u,v)}(g_0^{(1)}\varphi)\rangle &= e^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k} \\
&\quad\times\left(1+i\epsilon\Big(\sum_{k>0}\alpha_k\mathfrak{a}_{-k}-\bar{\alpha}_k\bar{\mathfrak{a}}_{-k}\Big)+i\epsilon vR(\alpha_{u,v}+\bar{\alpha}_{u,v})\right)\Omega_{u,v}|u,v\rangle \\
&= e^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k} \\
&\quad\times\left(1+i\epsilon\Big(\sum_{k>0}\alpha_k\mathfrak{a}_{-k}-\bar{\alpha}_k\bar{\mathfrak{a}}_{-k}\Big)+i\epsilon(h_{u,v}-\bar{h}_{u,v})\right)\Omega_{u,v}|u,v\rangle. \qquad (71)
\end{aligned}$$

The action of G_0 is somehow different. In this case, the $\bar{L}_0$ term does not annihilate $\langle u,v|$, but rather acts on it by multiplying by $\bar{h}_{u,v}$. We thus get

$$\begin{aligned}
G_0|\mathfrak{x}^{(u,v)}(\varphi)\rangle &= e^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k} \\
&\quad\times\left(1+i\epsilon\Big(\sum_{k>0}\alpha_k\mathfrak{a}_{-k}-\bar{\alpha}_k\bar{\mathfrak{a}}_{-k}\Big)+i\epsilon(h_{u,v}-\bar{h}_{u,v})\right)\Omega_{u,v}|u,v\rangle \\
&= |\mathfrak{x}^{(u,v)}(g_0^{(1)}\varphi)\rangle. \qquad (72)
\end{aligned}$$

This completes the proof.

6 Concluding remarks

This simple yet quite instructive calculation gives an example of a conformal theory with non-conformally invariant boundary conditions. Can the map $\varphi \to |\mathfrak{x}(\varphi)\rangle$ for the free boson be used to investigate minimal models with general boundary conditions? It is well known that minimal models can be constructed from the $c = 1$ CFT, using the Coulomb gas technique. This was succesfully done on the plane by Dotsenko and Fateev [6, 7] and on the torus by Felder [8]. This might be one path to construct the map for these models.

Langlands and the two authors have recently studied numerically the statistical distribution of the Fourier coefficients of a field defined for the Ising model. This distribution is more intricate than the boson's as the Fourier coefficients of the field at one boundary do not appear now to be mutually independent. The map $\varphi \to |\mathfrak{x}(\varphi)\rangle$, if it exists for the Ising model, might be a rich object.

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